Received: *November 18, 2000,* Accepted: *December 6, 2000*

BPS states in 10+2 dimensions

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Abstract: We discuss a (10+2)D $N = (1, 1)$ superalgebra and its projections to M-theory, type-IIA and -IIB algebras. From the complete classification of a second-rank central term valued in the so$(10, 2)$ algebra, we find all possible BPS states coming from this term. We show that, among them, there are two types of 1/2-susy BPS configurations; one corresponds to a super (2+2)-brane while another one arises from a nilpotent element in so$(10, 2)$.

Keywords: F-Theory, Extended Supersymmetry.

JHEP12(2000)006

Contents

1. Introduction

It is intriguing to consider the possibility that a theory of everything predicts not only the dimensionality of spacetime but also its signature. In a paper [1], with this spirit in mind, Blencowe and Duff investigated possible supersymmetric branes moving in spacetime without *a priori* assuming definite signatures of both the brane worldvolume and the spacetime. Requiring local κ-symmetry on the worldvolume and spacetime supersymmetry, they found, adding to well known string and membrane configurations, two extra octonionic type branes; a (1+2)-brane in (9+2)D and a (2+2)-brane in (10+2)D. In [2], the (1+2)-brane was shown to be realized as a solution of a (9+2)D theory obtained from the usual (10+1)D supergravity by spacelike and timelike T-duality transformations. The latter (2+2)-brane can possibly exist only when we consider an $N = (1,0)$ non-Poincaré spacetime supersymmetry, which is the maximal symmetry to have 32 real supercharges.

In recent development of string unification, the connection of various supersymmetric theories in higher dimensions has been discussed. It is well known that the conjectured M-theory, when compactified on S^1 and $S^1/\mathbb{Z}_2$, gives type IIA and $E_8 \times E_8$ heterotic string theories, respectively [3]. M-theory also leads to type-IIB, SO(32) heterotic and type I strings less directly upon compactification to nine dimensions. Other higher dimensional structures also have been explored to obtain a unified picture of those strings and to explain various duality relations among them, in which the notion of extra time dimensions has emerged as hidden dimensions of the higher dimensional unification theories [4, 5]. A notable example is F-theory [4] which has been proposed as a (10+2)D structure to give a geometrical explanation of the self S-duality of the type-IIB string. In [6], superalgebras in dimensions beyond eleven have been studied in the context of the unification of M-theory, type-IIA and -IIB algebras. With the restriction of the number of real supercharges to be 64 or less, they found two possible distinct superalgebras; $N = (2, 0)$ algebra in (10+2)D and $N = (1, 0)$ algebra in (11+3)D, where the latter is reduced to various algebras in lower dimensions; $N = 1$ algebras in (11+2)D, (10+3)D and (11+1)D, $N = 2$ algebra in (10+1)D and $N = (1, 1)$ algebra in (10+2)D.

A fundamental question about physics with two or more time dimensions is what kind of local theory can exist in such a spacetime. There have been many suggestions concerning it [7]-[15]. We should be careful to discuss the dynamical aspect of such a theory, since the concept of dynamics and energy used in ordinary theories with one time should be modified in the present case. Alternatively it would be a promising way to start with the analysis of the symmetrical aspect of the theory. Indeed, the structure of superalgebra is sensitive to the dimensionality and the signature of spacetime and would teach us about the possibility of extra time directions. It also seems possible to see what types of fundamental objects exist in the superalgebra through the investigation of central terms appearing in anti-commutators among supercharges.

In this paper, we discuss $N = (1, 1)$ and $(1, 0)$ superalgebras in (10+2)D, assuming that the algebras are relevant to some local theory in (10+2)D. As described in section 2, the $N = (1, 1)$ algebra is composed of the $SO(10, 2)$ generators M_{AB}, (pseudo)-Majorana spinor supercharges Q^α and tensorial central terms $Z^{(k)}_{A_1 \cdots A_k}$ $(k = 2, 3, 6, 7, 10, 11)$ [5, 7]. Since the algebra does not contain any vector generator, it is *not* the Poincaré type superalgebra, so is rather unfamiliar to us. However, we see that its Weyl projected $N = (1, 0)$ algebra is related to the (10+1)D M-theory algebra (M-algebra) and the (9+1)D type-IIA algebra by dimensional reduction. It is also shown that another projection of the (10+2)D $N = (1, 1)$ algebra, together with dimensional reduction to (9+1)D, leads to the type-IIB algebra.

A primary interest on the (10+2)D algebras is how many and what types of BPS configurations possibly exist in them. The above mentioned connection of the (10+2)D $N = (1, 1)$ algebra to M-, type-IIA and -IIB algebras may suggest a (10+2)D origin of various BPS states in M-theory and all known superstring theories. In this paper, we concentrate on the second-rank tensorial central term $Z^{(2)}_{AB}$ and investigate possible BPS states arising from this term. As a first step, we need to simplify the form of the central term. $Z^{(2)}$ belongs to the so$(10,2)$ algebra and hence transforms under the adjoint action of the SO$(10,2)$ group. Unlike the cases of complex or real compact Lie algebras, so$(10,2)$ elements are not reduced to a unique form under the SO$(10,2)$ action and hence the classification of so$(10,2)$ becomes complicated. With the use of results in [16, 17, 18], we classify conjugacy classes of so$(10,2)$ completely and construct representatives for the classes, which contain at most 6 parameters. The classification enables us to pick up all possible BPS states characterized by some fraction of surviving supersymmetry, as shown in section 3. It is also demonstrated how the (10+2)D BPS states reduce to those in M-theory and type-IIA and -IIB theories. Finally several concluding remarks are given in section 4.

The (10+2)D flat metric is $\eta_{AB} = (-,-,+,\ldots,+)$. $A, B, \ldots = \tilde{0}, 0, 1, \ldots, 10$ denotes the (10+2)D indices, while $a, b, \ldots = 0, 1, \ldots, 10$ and $\mu, \nu, \ldots = 0, 1, \ldots, 9$ are (10+1)D and (9+1)D indices, respectively.

2. $N = (1,1)$ superalgebra in (10+2)D

We consider a graded generalization of the SO$(10,2)$ Lorentz algebra with generators M_{AB} in terms of spinor supersymmetry generators Q^α,

$$[M_{AB}, M_{CD}] = M_{AC}\eta_{BD} + M_{BD}\eta_{AC} - M_{AD}\eta_{BC} - M_{BC}\eta_{AD}\,, \tag{2.1}$$

$$[M_{AB}, Q^\alpha] = -\frac{1}{2}(\Gamma_{AB})^\alpha{}_\beta Q^\beta\,. \tag{2.2}$$

With these commutation relations, we have to define anti-commutators among the Q^α's, which depends on the dimensionality and the signature of spacetime.

2.1 Spinors in $D = (S,T)$

In a general $D = (S,T)$ spacetime, with gamma matrices Γ_A,

$$\{\Gamma_A, \Gamma_B\} = 2\,\eta_{AB} I_D\,, \tag{2.3}$$

it can be shown that there exists a matrix B satisfying

$$\Gamma_A = \kappa B^{-1}\Gamma^*_A B\,, \qquad B^*B = \epsilon I_D\,, \tag{2.4}$$

where parameters κ and ϵ are specified by the value of $S - T$ [19]. See table 1. With the product of all time component gamma matrices $A = \Gamma^{0_1}\cdots\Gamma^{0_T}$, the charge conjugation matrix is defined as $C = \tilde{B}A$, which satisfies

$$\tilde{\Gamma}_A = (-1)^T\kappa\, C\Gamma_A C^{-1}\,, \qquad \tilde{C} = \epsilon\,\kappa^T(-1)^{T(T+1)/2}C\,, \tag{2.5}$$

$S-T \mod 8$	0, 1, 2	0, 6, 7	4, 5, 6	2, 3, 4
ϵ	$+1$	$+1$	-1	-1
κ	$+1$	-1	-1	$+1$

Table 1: Parameters κ and ϵ in (2.4)

where the tilde denotes transpose. In the $\epsilon = +1$ case, we can impose the Majorana ($\kappa = -1$) or the pseudo-Majorana ($\kappa = +1$) condition as

$$Q^* = BQ \quad \Longleftrightarrow \quad \bar{Q} \equiv Q^\dagger A = \tilde{Q} C \,. \tag{2.6}$$

In the even dimension D, we can define a definite chirality for spinors by the chirality matrix

$$\Gamma \equiv (-1)^{(S-T)/4} \Gamma^{0_1} \cdots \Gamma^{0_T} \Gamma^1 \cdots \Gamma^S \,, \qquad \Gamma^2 = I_D \,, \qquad \{\Gamma, \Gamma^A\} = 0 \,. \tag{2.7}$$

From (2.5) and (2.7), we obtain the condition $\Gamma = B^{-1}\Gamma^* B$, if $S - T = 0 \pmod 4$, showing that it is possible to impose the (pseudo)-Majorana-Weyl condition for spinors in 10+2 dimensions.

2.2 Anti-commutators

The general form of anti-commutation relations among the (pseudo-)Majorana spinors Q^α becomes

$$\{Q^\alpha, Q^\beta\} = \sum_k \frac{1}{k!} (\Gamma^{A_1 \cdots A_k} C^{-1})^{\alpha\beta} Z^{(k)}_{A_1 \cdots A_k} . \tag{2.8}$$

Since the left-hand-side (LHS) of (2.8) is a real symmetric matrix, we have to choose $Z^{(k)}$ which make the right-hand-side (RHS) of (2.8) symmetric. In 10+2 dimensions, using (2.5), we can show that

$$\begin{aligned} \Gamma^{A_1 \cdots A_k} C^{-1} \text{ is symmetric} \quad & \text{for } k = 1, 2 \mod 4 \text{ for the } C_- \text{ case,} \\ & \text{for } k = 2, 3 \mod 4 \text{ for the } C_+ \text{ case.} \end{aligned}$$

For the C_- case, the algebra consists of the set of generators $\{M_{AB}, Q^\alpha, Z^{(k)}\}$, ($k = 1, 2, 5, 6, 9, 10$), which becomes a supersymmetric extension of the Poincaré algebra if the vector $Z^{(1)}_A$ is identified with the translation generator P_A. This, however, causes a problem in the construction of the worldvolume theory [7]. Since the matrix $(\Gamma_{ABC} C_-^{-1})^{\alpha\beta}$ is anti-symmetric with respect to (α, β), the Wess-Zumino term on the (2+2)D worldvolume is identically zero, which means that the κ-symmetry can not be defined in this case. Thus the degrees of freedom of bosons and fermions on the worldvolume do not match up.

In the following, we consider the C_+ case which leads to a non-Poincaré superalgebra defined with (2.1), (2.2) and

$$\{Q^\alpha, Q^\beta\} = \frac{1}{2}\,(\Gamma^{AB}C_+^{-1})^{\alpha\beta} Z^{(2)}_{AB} + \sum_{k=3,6,7,10,11} \frac{1}{k!}(\Gamma^{A_1\cdots A_k}C_+^{-1})^{\alpha\beta} Z^{(k)}_{A_1\cdots A_k}\,, \tag{2.9}$$

where spinor indices α, β run from 1 to $2^{[12/2]} = 64$. The two-rank tensorial central term $Z^{(2)}_{AB}$ is supposed to be independent from the Lorentz generators M_{AB}. We note that $Z^{(2)}_{AB}$ has a constraint arising from the positivity condition that all eigenvalues λ_i of the matrix $\{Q, Q\}$ of (2.9) are non-negative. Henceforth, for simplicity, we take $C_+ = \Gamma^{\tilde{0}}\Gamma^0$ ($B_+ = I$) and the spinor Q^α to be real. Then it can be easily shown that the trace of all terms in the RHS of (2.9) vanishes except for the term $(\Gamma^{\tilde{0}0}C_+^{-1})Z_{\tilde{0}0}$,

$$\mathrm{Tr}\{Q^\alpha, Q^\beta\} = \sum_{i=1}^{64} \lambda_i = 64\, Z^{(2)}_{\tilde{0}0} \geq 0\,. \tag{2.10}$$

In the following, we demonstrate two projections of the (10+2)D spin space using the explicit representation of gamma matrices,

$$\Gamma^{\tilde{0}} = i\sigma_2 \otimes I_{32}\,, \qquad \Gamma^a = \sigma_1 \otimes \gamma^a\,, \tag{2.11}$$

where γ^a are (10+1)D gamma matrices, in which γ^μ equal to (9+1)D gamma matrices and $\gamma^{10} = \gamma^0 \cdots \gamma^9$,

$$\gamma^0 = i\sigma_2 \otimes I_{16}\,, \qquad \gamma^k = \sigma_1 \otimes \gamma^k_{(9)}\,, \quad (k = 1, \ldots, 9) \tag{2.12}$$

where $\gamma^k_{(9)}$ are 9D real symmetric gamma matrices and σ_i are Pauli matrices.

2.3 Weyl projection

We impose the Weyl condition to Q^α by the projection operator $P_\pm \equiv 1/2(I_{64} \pm \Gamma)$, where $\Gamma = \Gamma^{\tilde{0}}\Gamma^0 \cdots \Gamma^{10}$ in (2.7). In this projection, $Z^{(k)}$ with $k = 3, 7, 11$ drop out from (2.9) and anti-commutation relations for Majorana-Weyl spinors $Q^\alpha_\pm = (P_\pm)^\alpha{}_\beta Q^\beta$ become

$$\{Q^\alpha_\pm, Q^\beta_\pm\} = \frac{1}{2}(P_\pm\Gamma^{AB}C^{-1})^{\alpha\beta} T^{(\mp)}_{AB} + \frac{1}{6!}(\Gamma^{A_1\cdots A_6}C^{-1})^{\alpha\beta(\mp)} Z^{(6)}_{A_1\cdots A_6}\,, \tag{2.13}$$

where $T^{(\mp)}_{AB} = Z^{(2)}_{AB} \mp *Z^{(10)}_{AB}$ and $*Z^{(10)}_{AB}$ is the (10+2)D dual of $Z^{(10)}$. The tensor $Z^{(6)}$ becomes (anti-)self-dual under the projection. This is the $N = (1, 0)$ superalgebra in the (10+2)D flat spacetime, with 32 real supercharges [5, 7].

Let us consider the dimensional reduction to (10+1)D by the compactification of the $\tilde{0}$ time direction. We choose the projection P_- with the explicit realization (2.11), (2.12) of gamma matrices, which leads to the expression $Q^\alpha_- = (0, Q^{\tilde{\alpha}})$

($\tilde{\alpha} = 1, \ldots, 32$). With the redefinition of variables, $P_a = -T^{(+)}_{a\tilde{0}}$, $Y_{ab} = -T^{(+)}_{ab}$, $Z_{a_1\cdots a_5} = 2^{(+)}Z^{(6)}_{\tilde{0}a_1\cdots a_5}$, (2.13) becomes

$$\{Q^{\tilde{\alpha}}, Q^{\tilde{\beta}}\} = \left(\gamma^a C^{-1}_{(11)}\right)^{\tilde{\alpha}\tilde{\beta}} P_a + \frac{1}{2}\left(\gamma^{ab} C^{-1}_{(11)}\right)^{\tilde{\alpha}\tilde{\beta}} Y_{ab} + \frac{1}{5!}\left(\gamma^{a_1\cdots a_5} C^{-1}_{(11)}\right)^{\tilde{\alpha}\tilde{\beta}} Z_{a_1\cdots a_5}\,, \quad (2.14)$$

where $C_{(11)} = \gamma^0$ is the charge conjugation matrix in (10+1)D and $Q^{\tilde{\alpha}}$ is the (10+1)D Majorana spinor with respect to $C_{(11)}$. This is the (10+1)D M-theory superalgebra [21].

We further take the S^1 compactification of the spatial 10-direction of the (10+1)D spacetime. We decompose $Q^{\tilde{\alpha}}$ into two Majorana-Weyl supercharges $Q^{\tilde{\alpha}}_{A\pm}$ of opposite chirality with respect to the matrix γ^{10}. Acting the projector $P^{(10)}_{\pm} = \frac{1}{2}(I_{32} \pm \gamma^{10})$ on (2.14), we obtain the (9+1)D type-IIA superalgebra,

$$\begin{aligned}
\{Q^{\tilde{\alpha}}_{A\pm}, Q^{\tilde{\beta}}_{A\pm}\} =\;& (P^{(10)}_{\pm}\gamma^{\mu}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}(P_{\mu} \mp Z_{\mu}) + \frac{1}{5!}(\gamma^{\mu_1\cdots\mu_5}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}(\mp)}Z_{\mu_1\cdots\mu_5}\,,\\
\{Q^{\tilde{\alpha}}_{A\pm}, Q^{\tilde{\beta}}_{A\mp}\} =\;& \pm(P^{(10)}_{\pm}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}Z + \frac{1}{2}(P^{(10)}_{\pm}\gamma^{\mu\nu}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}Y_{\mu\nu} \pm \\
& \pm\frac{1}{4!}(P^{(10)}_{\pm}\gamma^{\mu_1\cdots\mu_4}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}Z_{\mu_1\cdots\mu_4}\,,
\end{aligned} \quad (2.15)$$

where $Z = P_{10}$, $Z_{\mu} = Y_{\mu 10}$, $Z_{\mu_1\cdots\mu_4} = Z_{10\mu_1\cdots\mu_4}$ and $C_{(10)} = C_{(11)} = \gamma^0$.

2.4 IIB projection

Let us take another projection of (2.9) by the projector $P_{B\pm} = 1/2(I_{64} \pm \hat{\Gamma})$ with $\hat{\Gamma} \equiv \Gamma^0\Gamma^1\cdots\Gamma^9$ [7]. The supercharges $Q^{\alpha}_{B\pm} = (P_{B\pm})^{\alpha}{}_{\beta}Q^{\beta}$ are not $\mathrm{SO}(10,2)$ covariant but covariant under $\mathrm{SO}(9,1)$. In this case, all terms in the RHS of (2.9) remain after the projection. We choose P_{B+} with the representation (2.11), (2.12), which becomes

$$P_{B+} = \begin{pmatrix} P^{(10)}_{+} & 0 \\ 0 & P^{(10)}_{+} \end{pmatrix}. \quad (2.16)$$

The projected Q^{α}_{B+} are written as $Q^{\alpha}_{B+} = (Q^{1\tilde{\alpha}}_{B}, Q^{2\tilde{\beta}}_{B})$, where two $Q^{i\tilde{\alpha}}_{B}$ have the same chirality with respect to γ^{10}. Then, compactifying timelike and spacelike $(\tilde{0}, 10)$-directions, we obtain the (9+1)D type-IIB superalgebra,

$$\begin{aligned}
\{Q^{i\tilde{\alpha}}_{B}, Q^{j\tilde{\beta}}_{B}\} =\;& \delta^{ij}(P^{(10)}_{+}\gamma^{\mu}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}(P_{\mu} - T_{\mu}) + (P^{(10)}_{+}\gamma^{\mu}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}\tilde{Z}^{ij}_{\mu} + \\
& + \frac{1}{3!}\epsilon^{ij}(P^{(10)}_{+}\gamma^{\mu_1\mu_2\mu_3}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}}T_{\mu_1\mu_2\mu_3} + \frac{1}{5!}\delta^{ij}(\gamma^{\mu_1\cdots\mu_5}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}(-)}Z_{\mu_1\cdots\mu_5} + \\
& + \frac{1}{5!}(\gamma^{\mu_1\cdots\mu_5}C^{-1}_{(10)})^{\tilde{\alpha}\tilde{\beta}(-)}\tilde{Z}^{ij}_{\mu_1\cdots\mu_5}\,,
\end{aligned} \quad (2.17)$$

where $P_\mu = Z^{(2)}_{\tilde{0}\mu}$, $T_\mu = *Z^{(10)}_{10\mu}$, $Z_{\mu_1\cdots\mu_5} = Z^{(6)}_{\tilde{0}\mu_1\cdots\mu_5}$ and

$$\begin{aligned}\tilde{Z}^{ij}_\mu &= (\sigma_3)^{ij}(Z^{(2)}_{10\mu} - *Z^{(10)}_{\tilde{0}\mu}) + (\sigma_1)^{ij}(*Z^{(11)}_\mu - Z^{(3)}_{\tilde{0}10\mu})\,,\\ T_{\mu_1\mu_2\mu_3} &= *Z^{(7)}_{\tilde{0}10\mu_1\mu_2\mu_3} - Z^{(3)}_{\mu_1\mu_2\mu_3}\,,\\ \tilde{Z}^{ij}_{\mu_1\cdots\mu_5} &= (\sigma_3)^{ij}Z^{(6)}_{10\mu_1\cdots\mu_5} + (\sigma_1)^{ij} * Z^{(7)}_{\mu_1\cdots\mu_5}\,.\end{aligned} \tag{2.18}$$

Note that the positivity condition $Z^{(2)}_{\tilde{0}0} \geq 0$ in (2.10) is translated into $P_0 \geq 0$ under the above two projections. Having clarified the relation of the (10+2)D $N =$ (1,1) algebra to M-theory, type-IIA and type-IIB algebras, in the next section we investigate the connection of BPS states inherent in the (10+2)D algebra with those in M-theory and string theories, starting from the classification of the (10+2)D BPS states.

3. (10+2)D BPS states

3.1 BPS states in (10+1)D M-algebra

Before classifying BPS states in (10+2)D, it is better to recall how to find BPS states in an ordinary Poincaré superalgebra. As an example, let us take the M-algebra in (10+1)D, with only the first term in the RHS of (2.14),

$$\{Q^{\tilde{\alpha}}, Q^{\tilde{\beta}}\} = (\gamma^a C^{-1}_{(11)})^{\tilde{\alpha}\tilde{\beta}} P_a\,. \tag{3.1}$$

A BPS state is annihilated by some combination of supercharges and hence has a configuration with $\det\{Q^\alpha, Q^\beta\} = 0$, which is equivalent to the null condition $P^2 = 0$. If P_a is null, by the action of the SO(10,1) rotation, we can choose a frame in which $P_a = (\lambda, \pm\lambda, 0, \ldots, 0)$, with a positive parameter λ. Then (3.1) becomes

$$\{Q^{\tilde{\alpha}}, Q^{\tilde{\beta}}\} = \lambda(I \pm \gamma_1\gamma_0)^{\tilde{\alpha}\tilde{\beta}}. \tag{3.2}$$

Since the matrix $\gamma_1\gamma_0$ squares to the identity and is also traceless, the half of eigenvalues of $\gamma_1\gamma_0$ are $+1$ and half -1. Thus we see that there exists a unique BPS state in (3.1) breaking half of the supersymmetries, which corresponds to a massless particle in (10+1)D [21]. We need the second and third central terms in the RHS of (2.14) to obtain other BPS states, e.g. M2 and M5 branes.

3.2 BPS states in (10+2)D superalgebra

3.2.1 Second-rank central term Z_{AB}

We now consider what types of BPS states exist in the (10+2)D $N = (1,1)$ superalgebra. In this paper, we concentrate on the anti-symmetric central term $Z^{(2)}_{AB}$ in the RHS of (2.9) and make other $Z^{(k)}$ zero;

$$\{Q^\alpha, Q^\beta\} = \frac{1}{2}(\Gamma^{AB}C^{-1})^{\alpha\beta} Z^{(2)}_{AB}\,. \tag{3.3}$$

In order to find possible BPS configurations inherent in (3.3), we intend to simplify the form of $Z^{(2)}_{AB}$ by the action of the SO(10, 2) group, just as the case of the translation generator P_a in the above M-algebra. The matrix $Z = (Z^{(2)A}{}_B)$ may be regarded as a map of the 12D vector space $V_{(12)}$ onto $V_{(12)}$; for $v = (v^A) \in V_{(12)}$, $Zv = (Z^{(2)A}{}_B v^B) \in V_{(12)}$. We see that Z belongs to the so(10, 2) algebra represented on $V_{(12)}$,

$$\mathrm{so}(10,2) = \{Z \in gl(12,R);\ \eta(Zu,v) + \eta(u,Zv) = 0\,,\ \text{for all } u,v \in V_{(12)}\}\,, \qquad (3.4)$$

where η is the (10+2)D metric $\eta(u,v) = \eta_{AB}u^A v^B$. An arbitrary element $Z \in \mathrm{so}(10,2)$ is transformed under the adjoint action of SO(10, 2); $Z \to \Lambda^{-1}Z\Lambda$, $\Lambda \in \mathrm{SO}(10,2)$. The problem to find BPS states then reduces, as a first step, to that of classifying all conjugacy classes in the so(10, 2) algebra under the adjoint action. In 12D euclidean space-time, under the SO(12) rotation, any Z in the so(12) algebra is reduced to canonical form which has six 2×2 anti-symmetric blocks $A_k = ih_k\sigma_2$ ($k = 1,\ldots,6$) on the diagonal part. However, in pseudo-euclidean space-time, some configurations of Z can not be brought to canonical form and we have to be careful for the classification.

It is known that an arbitrary element Z in a semisimple Lie algebra $\mathfrak{g}$ has a unique decomposition,

$$Z = S + N\,, \qquad [S,N] = 0\,. \qquad (3.5)$$

where S and N are semisimple and nilpotent elements in $\mathfrak{g}$, respectively.

An element $S \in \mathfrak{g}$ is called semisimple if its adjoint representation $ad(S)$ is a diagonalizable matrix. In complex or real compact Lie algebras, any semisimple element is reduced to a unique form like the above SO(12) case, since there exists a unique Cartan subalgebra in $\mathfrak{g}$. In the real non-compact case, however, there are several non-equivalent Cartan subalgebras under the adjoint group G. An example is the $sl(2,R)$ algebra, which has two Cartan algebras, $\{\lambda\sigma_3\}$ and $\{i\theta\sigma_2\}$ ($\lambda,\theta \in R$). These are obviously inequivalent, since the former generates a non-compact group, while the latter yields the 2D rotation. The classification of all Cartan subalgebras in real semisimple Lie algebras is given in a paper of Sugiura [16]. In the real non-compact case, there also appears a nilpotent element $N \in \mathfrak{g}$ such that $N^m \neq 0$ and $N^{m+1} = 0$ for some non-negative integer m. Thus, in our superalgebra defined on the pseudo-euclidean spacetime with two time directions, we expect to obtain several conjugacy classes in so(10, 2) containing nilpotent elements.

The complete classification of real semisimple Lie algebras under the action of its adjoint group is given in a paper of Burgoyne and Cushman [18]. In the following, we classify the so(10, 2) algebra according to the method in [18] and construct a representative for each conjugacy class in so(10, 2).

3.2.2 Types Δ

We give a brief review on results in [18] which are needed in the rest of this paper. We start with the general linear group $\mathrm{GL}(12,C) = \mathrm{GL}(V_{(12)})$ represented on the

12D complex vector space $V_{(12)}$. The orthogonal group $\mathrm{O}(10,2)$ is defined as a real form of $\mathrm{GL}(12,C)$ specified by an automorphism σ of $\mathrm{GL}(12,C)$ with $\sigma^2=+1$,

$$\mathrm{O}(10,2)=\{g\in \mathrm{GL}(12,C)\,|\,\eta(gu,gv)=\eta(u,v) \text{ for all } u,v\in V_{(12)}\,,\ \sigma^{-1}g\sigma=g\}\,. \tag{3.6}$$

Note that σ acts on $V_{(12)}$ as an anti-linear map onto $V_{(12)}$ with $\eta(\sigma u,\sigma v)=\eta(u,v)^*$. The corresponding Lie algebra of $\mathrm{O}(10,2)$ is $\mathrm{o}(10,2)=\mathrm{o}(V_{(12)},\eta,\sigma)\simeq \mathrm{so}(10,2)$, which is given in (3.4).

We introduce an equivalence relation among pairs of the form $(Z,V_{(n)})$, where $Z\in o(V,\tau_{(p)},\sigma)$. The algebra $\mathrm{o}(V,\tau_{(p)},\sigma)$ is associated with the group $\mathrm{O}(n-p,p)$ and represented on $V_{(n)}$ with the metric $\tau_{(p)}$. Let $Z'\in \mathrm{o}(V'_{(n)},\tau'_{(p)},\sigma')$ then we write $(Z,V_{(n)})\sim(Z',V'_{(n)})$ if there exists an isomorphism ϕ of $V_{(n)}$ onto $V'_{(n)}$ such that $\phi Z=Z'\phi$, $\phi\sigma=\sigma'\phi$ and $\tau_{(p)}(u,v)=\tau'_{(p)}(\phi u,\phi v)$ for $u,v\in V_{(n)}$. In [18], an equivalence class for $\sim$ is called a *type* Δ and its dimension is given by $\dim\Delta=\dim V_{(n)}=n$ if $(Z,V_{(n)})\in\Delta$. The index of Δ denoted as $\mathrm{ind}\,\Delta$ is the number of time (negative sign) components in the metric $\tau_{(p)}$ in $\mathrm{o}(V_{(n)},\tau_{(p)},\sigma)$, i.e. $\mathrm{ind}\,\Delta=p$. The type Δ is nothing but a conjugacy class of the $\mathrm{o}(n-p,p)$ algebra under the adjoint action of $\mathrm{O}(n-p,p)$:

Proposition 1. ([18, section 2.1]) *Let $Z,X\in o(V_{(n)},\tau_{(p)},\sigma)$. There exists a $g\in O(V_{(n)},\tau_{(p)},\sigma)$ such that $g^{-1}Zg=X$ if and only if $(Z,V_{(n)})$ and $(X,V_{(n)})$ belong to the same type.*

Next, we introduce the notion of indecomposable type. Let $Z\in o(V_{(n)},\tau_{(p)},\sigma)$ and let Δ denote the type containing $(Z,V_{(n)})$. We suppose that $V_{(n)}=W_1+W_2$ is a sum of proper, disjoint, Z-invariant, σ-invariant, and orthogonal subspaces. Let $\tau^i_{(p)}$ $(i=1,2)$ be the restriction of the metric $\tau_{(p)}$ to each W_i, then the restriction of Z to each W_i denoted as Z_i belongs to $\mathrm{o}(W_i,\tau^i_{(p)},\sigma)$. Let Δ_i denote the type containing (Z_i,W_i). Then we write $\Delta=\Delta_1+\Delta_2$. A type Δ is called *indecomposable* if it can not be written as the sum of two or more types. For any type Δ, we have

Theorem. ([18, section 2.2]) *The decomposition $\Delta=\Delta_1+\Delta_2+\cdots+\Delta_s$ into indecomposable types is unique.*

In the decomposition, a representative Z in Δ can be given as a matrix to have representatives Z_k in Δ_k $(k=1,\ldots,s)$ on its diagonal part, with respect to the metric $\tau_{(p)}$ which has metrices $\tau^k_{(p)}$ in Δ_k on its diagonal. If a type Δ is decomposed as above, for the dimension and the index of Δ, we have

$$\begin{aligned}\dim\Delta&=\dim\Delta_1+\dim\Delta_2+\cdots+\dim\Delta_s\,,\\ \mathrm{ind}\,\Delta&=\mathrm{ind}\,\Delta_1+\mathrm{ind}\,\Delta_2+\cdots+\mathrm{ind}\,\Delta_s\,.\end{aligned} \tag{3.7}$$

In table 2, we write down all indecomposable types and their dimension and index for the $\mathrm{o}(q,p)$ algebra. In our $\mathrm{O}(10,2)$ case, all types Δ, i.e., all conjugacy classes

Type			$\dim\Delta$	$\mathrm{ind}\,\Delta$
$\Delta_m(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta\neq\pm\bar\zeta$		$4(m+1)$	$2(m+1)$
$\Delta_m(\zeta,-\zeta)$	$\zeta=\bar\zeta\neq 0$		$2(m+1)$	$m+1$
$\Delta^\epsilon_m(\zeta,-\zeta)$	$\zeta=-\bar\zeta\neq 0$	m: even	$2(m+1)$	$m+1-(-1)^{\frac{m}{2}}\epsilon$
		m: odd	$2(m+1)$	$m+1$
$\Delta^\epsilon_m(0)$		m: even	$m+1$	$\frac{1}{2}(m+1-(-1)^{\frac{m}{2}}\epsilon)$
$\Delta_m(0,0)$		m: odd	$2(m+1)$	$m+1$

Table 2: Indecomposable types in the $\mathrm{o}(q,p)$ algebra

in the $\mathrm{o}(10,2)$ algebra can be obtained by taking all possible combinations of the indecomposable types with the conditions $\dim\Delta=12$ and $\mathrm{ind}\,\Delta=2$, which will be done in the following sections.

We briefly explain the notation of indecomposable types in table 2 and how to construct a representative for each type. Henceforth we omit the subscripts of $V_{(n)}$ and $\tau_{(p)}$ for simplicity. Let $(Z,V)\in\Delta_m(\zeta_1,\dots,\zeta_t)$, where the subscript $m=0,1,\dots$ means that, in the decomposition $Z=S+N$ in section 3.2.1, the nilpotent N satisfies $N^m\neq 0$ and $N^{m+1}=0$, while parameters $\zeta_1,\dots,\zeta_t\in C$ are eigenvalues of the semisimple element S. Let F be a subspace of the vector space V spanned by eigenvectors $\{e_1,\dots,e_t\}$ of S. We can show that V is decomposed as $V=F+NF+\cdots+N^mF$. Now the semisimple S can be supposed to belong to the algebra $\mathrm{o}(F,\tau_m,\sigma)$ represented on F with the metric $\tau_m(u,v)\equiv\tau(u,N^mv)$ for $u,v\in F$. Note that, from the nilpotency of N, the metric $\tau_m(u,v)=0$ for $u,v\in NV$. Non-zero values of τ_m are given for each indecomposable type in the [18, appendix 2]. The description of the indecomposable types enables us to construct an explicit matrix realization of $Z=S+N\in o(V,\tau,\sigma)$ and the metric τ on V. In particular, descriptions of Z and τ as real matrices on V can be given easily by taking σ-invariant combinations of elements in V. In the appendix A in this paper, we give explicit forms of representatives for all indecomposable types relevant to our $\mathrm{O}(10,2)$ case. Having the representatives and arranging them on the diagonal part of 12×12 matrix, we obtain a realization of a representative for each type in $\mathrm{o}(10,2)$, which is with respect to a (10+2)D metric with metrices of indecomposable types arranged on its diagonal part. Finally, by an isomorphism ϕ of V onto V noted above, we have our seeking forms of representatives with respect to our (10+2)D metric $\eta=\mathrm{diag}(-,-,+\cdots,+)$.

As a final step, we move to the classification of the $\mathrm{so}(10,2)$ algebra under the action of the $\mathrm{SO}(10,2)$ group, rather than $\mathrm{O}(10,2)$. More exactly, we classify $\mathrm{so}(10,2)$ under the subgroup of $\mathrm{SO}(10,2)$ generated by the $\mathrm{so}(10,2)$ generators M_{AB}, which we denote as $\mathrm{SO}(10,2)_0$. The subgroup is the connected part of $\mathrm{SO}(10,2)$ containing the identity element and its action to an arbitrary element Z in $\mathrm{so}(10,2)$ does not change the sign of the component $Z_{\tilde{0}0}$ in (2.10). The quotient group $\mathrm{O}(10,2)/\,\mathrm{SO}(10,2)_0$ is equal to $Z_2\times Z_2$. One of Z_2's corresponds to the sign of determinant, while another

Z_2 indicates that SO$(10,2)$, or more generally SO(q,p), has two connected parts. The two connected parts are distinguished by the sign of the principal minor of the time components of $\Lambda \in$ SO(q,p) [22]. In SO$(10,2)$, we have the condition $(\Lambda^{\tilde{0}}{}_{\tilde{0}}\Lambda^{0}{}_{0} - \Lambda^{0}{}_{\tilde{0}}\Lambda^{\tilde{0}}{}_{0})^2 \geq 1$, where the positive sign of the determinant corresponds to SO$(10,2)_0$. We choose an element D (C) in O$(10,2)$ which has the negative (positive) determinant and the positive (negative) principal minor for the time components. If the action of C and D on a representative Z in a type Δ in O$(10,2)$,

$$\begin{aligned} &{}^{(+,+)}Z = Z\,, \qquad {}^{(+,-)}Z = C^{-1}Z\,C\,, \qquad {}^{(-,+)}Z = D^{-1}Z\,D\,, \\ &{}^{(-,-)}Z = (CD)^{-1}Z\,(CD)\,, \end{aligned} \tag{3.8}$$

is non-trivial, then the type Δ splits into four distinct types ${}^{(\epsilon_D,\epsilon_C)}\Delta \ni {}^{(\epsilon_D,\epsilon_C)}Z$, $(\epsilon_D, \epsilon_C = \pm 1)$ under SO$(10,2)_0$. Some Δ may split into only two types ${}^{(\epsilon_D,+)}\Delta$ $({}^{(+,\epsilon_C)}\Delta)$ if Z in Δ is invariant under the C (D) action, while some Δ may be invariant under SO$(10,2)_0$.

With the description noted above, we give all types Δ, i.e. conjugacy classes of the so$(10,2)$ algebra under the SO$(10,2)_0$ action and find possible BPS configurations coming from the types.

3.3 so$(2,2)$ case

3.3.1 so$(2,2)$ conjugacy classes

Before studying the full so$(10,2)$ case, we demonstrate the classification of its so$(2,2)$ part. We will see that almost possible BPS states arise from this part. For the SO$(2,2)$ group, it is known that its associated Lie algebra has the decomposition so$(2,2) \simeq sl(2,R)_L \times sl(2,R)_R$. We take the basis $\{H_i, E_i, F_i\}$ for each $sl(2,R)_i$ satisfying

$$[H_i, E_i] = E_i\,, \qquad [H_i, F_i] = -F_i\,, \qquad [E_i, F_i] = 2H_i\,, \tag{3.9}$$

with an explicit realization of them,

$$H_L = \frac{1}{2}\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad H_R = \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{3.10}$$

and

$$E_L = \frac{1}{2}\begin{pmatrix} \varepsilon_2 & -\varepsilon_2 \\ \varepsilon_2 & -\varepsilon_2 \end{pmatrix}, \qquad E_R = \frac{1}{2}\begin{pmatrix} \varepsilon_2 & -\sigma_1 \\ -\sigma_1 & \varepsilon_2 \end{pmatrix}, \tag{3.11}$$

and $F_L = \tilde{E}_L$ and $F_R = \tilde{E}_R$, where the 2×2 real matrix $\varepsilon_2 = -i\sigma_2$. For later use, we define

$$\begin{aligned} K_L &= E_L - F_L = \begin{pmatrix} \varepsilon_2 & 0 \\ 0 & -\varepsilon_2 \end{pmatrix}, \\ K_R &= E_R - F_R = \begin{pmatrix} \varepsilon_2 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}. \end{aligned} \tag{3.12}$$

For the C- and D-splittings noted in section 3.2.2 we use

$$C = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad D = \begin{pmatrix} I_2 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{3.13}$$

In the following, we show representatives for all conjugacy classes of the so$(2,2)$ algebra with respect to the (2+2)D metric $\eta_{AB} = (-,-,+,+)$, $(A, B = \tilde{0}, 0, 9, 10)$ in terms of $\{H_i, E_i, F_i\}$ and K_i.

(I) Semisimple cases.

$$\begin{aligned}
&\text{(i) } \mathcal{S}_1 = (h_1+h_2)H_1 + (h_1-h_2)H_2 \ \in {}^{(\pm,+)}(\Delta_0(h_1,-h_1) + \Delta_0(h_2,-h_2)), \\
&\qquad (h_1, h_2 \in \mathbb{R},\ h_1 \geq 0,\ |h_2| \leq 0) && (3.14) \\
&\text{(ii) } \mathcal{S}_2 = \frac{1}{2}(h_1+h_2)K_L + \frac{1}{2}(h_1-h_2)K_R \ \in {}^{(\pm,\pm)}(\Delta_0^-(-ih_1, ih_1) + \Delta_0^+(ih_2,-ih_2)), \\
&\qquad (h_1, h_2 \in \mathbb{R}) && (3.15) \\
&\text{(iii) } \mathcal{S}_3 = 2h_1H_L + h_2K_R \ \in {}^{(+,\pm)}\Delta_0(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta}), \\
&\qquad (\zeta = h_1 + ih_2,\ h_1 \geq 0,\ h_2 \in \mathbb{R}) && (3.16) \\
&\text{(iv) } \mathcal{S}_4 = h_2K_L + 2h_1H_R \ \in {}^{(-,\pm)}\Delta_0(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta}), \\
&\qquad (\zeta = h_1 + ih_2,\ h_1 \geq 0,\ h_2 \in \mathbb{R}). && (3.17)
\end{aligned}$$

In (I-i), $\epsilon_D = +1$ (-1) corresponds to the positive (negative) h_2. In (I-ii), four distinct types labeled by $(\pm,\pm)$ correspond to four quadrants in the (h_1, h_2) plane. The sign $\epsilon_C = +1$ (-1) is for the positive (negative) h_2 in (I-iii) and (I-iv).

(II) Nilpotent cases.

$$\begin{aligned}
&\text{(i) } \mathcal{N}_{1\pm} = \pm(E_L + E_R) \ \in {}^{(+,\pm)}(\Delta_2^+(0) + \Delta_0^+(0)), && (3.18) \\
&\text{(ii) } \mathcal{N}_{2\pm} = \pm(E_L - E_R) \ \in {}^{(\mp,+)}(\Delta_0^-(0) + \Delta_2^-(0)), && (3.19) \\
&\text{(iii) } \mathcal{N}_{3\pm} = \pm E_L \ \in {}^{(+,\pm)}\Delta_1(0,0)\,, && (3.20) \\
&\text{(iv) } \mathcal{N}_{4\pm} = \pm E_R \ \in {}^{(-,\pm)}\Delta_1(0,0)\,. && (3.21)
\end{aligned}$$

(III) Semisimple + nilpotent cases.

$$\begin{aligned}
&\text{(i) } \mathcal{M}_{1\pm} = \quad hH_L \pm E_R \ \in {}^{(-,\pm)}\Delta_1(-h/2, h/2)\,, && (h > 0) && (3.22) \\
&\text{(ii) } \mathcal{M}_{2\pm} = \pm E_L + hH_R \ \in {}^{(+,\pm)}\Delta_1(-h/2, h/2)\,, && (h > 0) && (3.23) \\
&\text{(iii) } \mathcal{M}_{3\pm} = \pm E_L + hK_R \ \in {}^{(-,\mp)}\Delta_1^+(ih, -ih)\,, && (h > 0) \\
&\qquad\qquad \in {}^{(+,\mp)}\Delta_1^-(ih, -ih)\,, && (h < 0) && (3.24) \\
&\text{(iv) } \mathcal{M}_{4\pm} = hK_L \pm E_R \ \in {}^{(+,\mp)}\Delta_1^+(ih, -ih)\,, && (h > 0) \\
&\qquad\qquad \in {}^{(+,\mp)}\Delta_1^-(ih, -ih)\,, && (h < 0)\,. && (3.25)
\end{aligned}$$

3.3.2 BPS states

Having obtained all inequivalent representatives in so(2, 2), we substitute them into the RHS of (3.3) and calculate the characteristic polynomial,

$$D(x) = \det(\{Q, Q\} - xI_{64}), \tag{3.26}$$

for the 64×64 matrix $\{Q^\alpha, Q^\beta\}$. We also denote the characteristic polynomial for the Weyl-projected 32×32 matrix $\{Q^{\tilde{\alpha}}, Q^{\tilde{\beta}}\}$ as $D_W(x)$. Note that eigenvalues of $\{Q, Q\}$ do not depend on the choice of a particular representative of types in so(2, 2). As mentioned in section 2, by the positivity condition, the eigenvalues must be non-negative. In the following, we show only the cases which satisfy the positivity condition.

[I] $\quad \mathcal{S}_2 = \frac{1}{2}(h_1 + h_2)K_L + \frac{1}{2}(h_1 - h_2)K_R$

With the form of $\mathcal{S}_2$, (3.3) becomes

$$\{Q^\alpha, Q^\beta\} = h_1 I_{64} - h_2 \Gamma^{\tilde{0}0910}, \tag{3.27}$$

with its characteristic polynomial,

$$D(x) = [D_W(x)]^2, \qquad D_W(x) = [x - (h_1 + h_2)]^{16}[x - (h_1 - h_2)]^{16}, \tag{3.28}$$

where $h_1 = Z_{\tilde{0}0} \geq 0$. The bound $h_1 \geq |h_2|$ is imposed by the positivity condition. A BPS state arises if and only if the bound is saturated, i.e. $h_1 = |h_2|$, in which the fraction of survived supersymmetries is 1/2. In this case, the eigenspinor ϵ of $\{Q^\alpha, Q^\beta\}$ with zero eigenvalue satisfies

$$\Gamma^{\tilde{0}0910}\epsilon = \pm\epsilon, \tag{3.29}$$

where the sign $+(-)$ is for positive (negative) h_2. From (3.29), it is natural to interpret the $\mathcal{S}_2$ case as the configuration of an extended object in the $(\tilde{0}, 0, 9, 10)$ directions, which we call super (2+2)-brane. We note that $\mathcal{S}_3$ and $\mathcal{S}_4$ cases with the positivity condition reduce to the BPS saturated configuration in the $\mathcal{S}_2$ case.

Let us check how the (2+2)-brane is reduced to other branes in M-theory and type-IIA theory by the dimensional reduction described in section 2.3. The condition (3.29) with, say, the minus sign in the RHS becomes under the Weyl projection,

$$\gamma^{0910}\epsilon_- = \epsilon_-, \tag{3.30}$$

where ϵ_- is a chiral part of ϵ with $\Gamma\epsilon_- = -\epsilon_-$. (3.30) shows that the dimensionally reduced super (2+2)-brane is nothing but a supermembrane extended in the $(0, 9, 10)$ directions. The further dimensional reduction of the spatial 10-direction yields the type-IIA fundamental string configuration in the $(0, 9)$ directions, characterized by the condition

$$\gamma^{09}\epsilon_L = \epsilon_L, \qquad \gamma^{09}\epsilon_R = -\epsilon_R, \tag{3.31}$$

where $\epsilon_{L(R)}$ is a chiral part of ϵ_- with chirality +1 (−1) with respect to the (9+1)D chirality matrix γ^{10}.

On the other hand, under dimensional reduction to (9+1)D with the IIB projection in section 2.4, (3.29) becomes

$$\gamma^{09}\epsilon_L = \epsilon_L\,, \qquad \gamma^{09}\epsilon_R = -\epsilon_R\,, \tag{3.32}$$

where the eigenspinor ϵ is decomposed as $\epsilon = (\epsilon_R, \epsilon_L)$ with $\gamma^{10}\epsilon_{L(R)} = \epsilon_{L(R)}$. It is known that (3.32) can be derived from (3.31), upon compactification to (8+1)D, by the T-duality transformation and corresponds to the (9+1)D type-IIB fundamental string.

[II] $\quad \mathcal{N}_{1+} = E_L + E_R$
In this nilpotent case, $Z_{\tilde{0}0} = Z_{09} = 1$ and the others zero. Then (3.3) takes the form,

$$\{Q^\alpha, Q^\beta\} = I_{64} + \Gamma^{9\tilde{0}}\,, \tag{3.33}$$

with

$$D(x) = [D_W(x)]^2\,, \qquad D_W(x) = x^{16}(x-2)^{16}\,. \tag{3.34}$$

This $\mathcal{N}_{1+}$ case also yields a BPS state with 1/2 surviving supersymmetries. The BPS state, however, seems rather different from the one in the $\mathcal{S}_2$ case, since, from (3.34), the former is definitely BPS saturated without any non-BPS excited configuration. It is caused by the fact that the nilpotent Z_{AB} is parameter independent.

Let us see how the BPS state is observed after the dimensional reduction to M-theory. (3.34) indicates that the reduced state in M-theory is still BPS with 1/2-susy, which has a non-zero Y_{09} charge in the algebra (2.14). In [20, 21], it was argued that the dual of Y_{09} corresponds to a charge of the M9 brane extended in $(0, 1, \ldots, 8, 10)$ directions. Subsequently, in [23], the target space solution for the M9 brane was found to be a domain wall solution of the massive (10+1)D supergravity.

Under another dimensional reduction to (9+1)D, the (10+2)D BPS state does not reduce to any BPS state in the type-IIB theory. In this reduction, $Z_{\tilde{0}0}$ becomes the energy P_0 in (9+1)D, while Z_{09} does not appear in the algebra (2.17). Hence the reduced state is interpreted as a non-BPS massive particle in the type-IIB theory.

[III] $\quad \mathcal{M}_{3+} = E_L + hK_R$
The characteristic polynomial in this semisimple+nilpotent case becomes

$$D(x) = [D_W(x)]^2\,, \qquad D_W(x) = x^8(x-2h)^{16}(x-2)^8\,. \tag{3.35}$$

The eigenvalues of the matrix $\{Q, Q\}$ are $x = 0, 2, 2h$ and thus the parameter h must be positive. As the number of zero eigenvalues is 8, this $\mathcal{M}_{3+}$ case corresponds to a BPS state with 1/4-susy. In the case of $h = 0$, which is the nilpotent $\mathcal{N}_{3+}$ case, there arise 16 additional zero eigenvalues and the fraction of surviving supersymmetries becomes 3/4.

BPS state	Susy	M-susy	IIA-susy	IIB-susy
$\mathcal{S}_2$ = (2+2)-brane	1/2	1/2	1/2	1/2
$\mathcal{N}_{1+} = E_L + E_R$	1/2	1/2	1/2	no-susy
$\mathcal{M}_{3+} = E_L + hK_R$	1/4	1/4	1/4	no-susy
$\mathcal{N}_{3+} = E_L$	3/4	3/4	3/4	1/2
$\mathcal{M}_{4+} = hK_L + E_R$	1/4	1/4	1/4	no-susy
$\mathcal{N}_{4+} = E_R$	3/4	3/4	3/4	1/2

Table 3: BPS states arising from the so$(2,2)$ part of so$(10,2)$

These fractions 1/4 and 3/4 remain unchanged under the Weyl projection. The $\mathcal{M}_{3+}$ case reduces to the 1/4-susy configuration with $P_0 = h + 1/2$, $P_{10} = -1/2$, $Y_{09} = -1/2$, $Y_{910} = h - 1/2$ and the others zero in the M-algebra (2.14). Since, as noted in [I], [II], Y_{910} and Y_{09} are identified with M2 and M9 brane charges, it would be possible to interpret the configuration as a composite state of these branes. It is needed to investigate whether the interpretation is also applicable to the $\mathcal{N}_{3+}$ case, or not.

Under the type-IIB projection, the $\mathcal{M}_{3+}$ case reduces to the IIB state with $P_0 = h + 1/2$, $\tilde{Z}_9^{11} = -\tilde{Z}_9^{22} = h - 1/2$ and the others zero in the algebra (2.17). It is obvious that the BPS bound in the type-IIB theory can not be saturated unless the parameter h equals to zero. Thus the 1/4-susy BPS state in the $\mathcal{M}_{3+}$ case corresponds to a non-BPS string state, while the $\mathcal{N}_{3+}$ case gives a 1/2-susy BPS string state.

[IV] $\mathcal{M}_{4+} = hK_L + E_R$
This case and the $\mathcal{N}_{4+} = E_R$ case can be obtained by exchanging the subscripts $L \leftrightarrow R$ in [III]. All of results on possible BPS states in the $\mathcal{M}_{4+}$ ($\mathcal{N}_{4+}$) case are the same as ones in the $\mathcal{M}_{3+}$ ($\mathcal{N}_{3+}$) case, except that the signs of $Z_{\tilde{0}10}$ and Z_{910} are opposite to those in [III]. Hence, through the Weyl projection, we have 1/4 and 3/4-susy states in [III] with the opposite signs of P_{10} and Y_{910}, while, through the IIB projection, we obtain string states in [III] with the opposite sign of $\tilde{Z}_9^{ii}$.

The above $\mathcal{S}_2$, $\mathcal{N}_{3+}$ and $\mathcal{N}_{4+}$ cases were previously discussed in [24]. In summary, we give the list of all possible BPS states in table 3.

3.4 so$(10,2)$ case

Finally we show the classification of the full so$(10,2)$ case. All types Δ in so$(10,2)$ can be divided into two cases; [I] the so$(2,2)$ part in section 3.3 plus the remaining so(8) part and [II] other cases which can not be decomposed as in [I].

[I] so$(2,2)$ part + so(8) part
Since the group SO(8) is compact, all types Δ in this case are given by the sum of

types in so$(2,2)$ and the so(8) type with the canonical form as its representative,

$$\Delta_{\mathrm{so}(10,2)} = \Delta_{\mathrm{so}(2,2)} + \sum_{k=3}^{6} \Delta_0^+(ih_k, -ih_k), \tag{3.36}$$

where $\dim \Delta_{\mathrm{so}(10,2)} = 12$ and $\mathrm{ind}\, \Delta_{\mathrm{so}(10,2)} = 2$, as $\dim \Delta_0^+ = 2$ and $\mathrm{ind}\, \Delta_0^+ = 0$. A representative for each type $\Delta_{\mathrm{so}(10,2)}$ is therefore obtained by arranging a representative of each $\Delta_{\mathrm{so}(2,2)}$ and four 2×2 anti-symmetric blocks $ih_k\sigma_2$ $(k = 3, 4, 5, 6)$ on the diagonal part.

[I-1] $\quad \mathcal{S}_2(h_1, h_2) + \sum_{k=3}^{6} ih_k\sigma_2$

This case with $\mathcal{S}_2$ in section 3.3 corresponds to the (10+2)D canonical form and is naturally interpreted as the configuration of super (2+2)-branes extended in timelike $(\tilde{0}, 0)$- and spacelike (1,2)-, (3,4)-, (5,6)-, (7,8)-, (9,10)-directions. The calculation of the characteristic polynomial gives

$$D(x) = \prod_{\pm}^{2^5} (h_1 \pm h_2 \pm h_3 \pm h_4 \pm h_5 \pm h_6 - x)^2 \,, \tag{3.37}$$

where $h_1 = Z_{\tilde{0}0} \geq 0$ and the above expression means the product of all possible combinations of the signs $\pm$. Since the total number of combinations equals to $2^5 = 32$, the RHS of (3.37) becomes a polynomial of order 64 of the parameter x, as expected.

It is easy to evaluate all possible fractions of survived supersymmetries in (3.37). At first, let parameters h_i $(i = 2, \ldots, 6)$ be non-zero. Then the unique minimal eigenvalue among $x = h_1 \pm h_2 \pm h_3 \pm h_4 \pm h_5 \pm h_6$ is

$$x_{min.} \equiv h_1 - |h_2| - |h_3| - |h_4| - |h_5| - |h_6| \,, \tag{3.38}$$

and thus, from the positivity condition, we have the bound,

$$h_1 \geq |h_2| + |h_3| + |h_4| + |h_5| + |h_6| \,. \tag{3.39}$$

A BPS state arises when the bound is saturated and, as each eigenvalue enters twice in the RHS of (3.37), the fraction of surviving supersymmetry is 1/32. Next, we suppose that one of parameters, say h_6, is zero. In this case, two eigenvalues $x_{\mathrm{mim.}}$ and $x_{\mathrm{mim.}} + 2|h_6|$ become minimal and hence we have a BPS state with 1/16-susy. More generally, we obtain a 1/32, 1/16, 1/8, 1/4, 1/2-susy BPS state if the number of zero h_i's is $0, 1, 2, 3, 4$, respectively.

BPS states in the $N = (1, 0)$ superalgebra can be found by taking the Weyl projection of $D(x)$,

$$D_W(x) = \prod_{-:\mathrm{even}}^{2^4} (h_1 \pm h_2 \pm h_3 \pm h_4 \pm h_5 \pm h_6 - x)^2 \,, \tag{3.40}$$

where all combinations with even number of negative signs are taken in the RHS. Here, we should be careful for finding zero eigenvalues, since the minimal value $x_{\rm min.}$ in (3.38) may or may not be contained in the RHS of (3.40). The complete classification of BPS states in this Weyl-projected case has been done in [25].

If one of h_i $(i = 2, \ldots, 6)$ is zero, $x_{\rm min.}$ always belongs to the set of eigenvalues in (3.40) and other eigenvalues are obtained by adding $2|h_{i_1}| + 2|h_{i_2}|$ or $2\sum_{q=1}^{4}|h_{i_q}|$ to $x_{\rm min.}$. As in the case (3.37), we can easily count the number of eigenvalues equal to $x_{\rm min.}$ and obtain a BPS state with 1/16, 1/8, 1/4, 1/2-susy, when the number of zero h_i's is $1, 2, 3, 4$, respectively. On the other hand, let us suppose all h_i to be non-zero. If $x_{\rm min.}$ is contained in (3.40), then $x_{\rm min.}$ is the unique minimal eigenvalue and its associated BPS state has the 1/16-susy. If $x_{\rm min.}$ is not in (3.40), the minimal eigenvalue in (3.40) is given by adding some $+2|h_k|$ to $x_{\rm min.}$, where the parameter h_k is supposed to have a minimal absolute value among h_i's. If there are q h_i's with the minimal absolute value, the fraction of survived supersymmetry becomes $q/16$ $(q = 1, \ldots, 5)$ when the BPS bound is saturated. In summary, in the (10+2)D N=(1,0) superalgebra, there arise 1/16, 1/8, 3/16, 1/4, 5/16 and 1/2-susy BPS states from the second-rank central term.

[I-2] Other $\text{so}(2,2)$ cases $+ \sum_{k=3}^{6} ih_k\sigma_2$

There also appear types Δ in so(10, 2) with $\mathcal{N}_{1+}$, $\mathcal{M}_{3+}$, $\mathcal{N}_{3+}$, $\mathcal{M}_{4+}$ and $\mathcal{N}_{4+}$ in the so(2, 2) part. Characteristic polynomials for the types become

$$\begin{aligned}
\mathcal{N}_{1+}:\ & D(x) = (D_W(x))^2\,, \\
& D_W(x) = \prod_{\pm}^{8}(x(x-2) - (h_3 \pm h_4 \pm h_5 \pm h_6)^2)^2\,,
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
\mathcal{M}_{3+}:\ & D(x) = D'_W(x) D_W(x)\,, \\
& D_W(x) = \prod_{-:\text{odd}}^{4}(x(x-2) - (h_3 \pm h_4 \pm h_5 \pm h_6)^2)^2 \times \\
& \qquad \times \prod_{-:\text{even}}^{8}(x - 2h \pm h_3 \pm h_4 \pm h_5 \pm h_6)^2\,, \\
& D'_W(x) = \prod_{-:\text{even}}^{4}(x(x-2) - (h_3 \pm h_4 \pm h_5 \pm h_6)^2)^2 \times \\
& \qquad \times \prod_{-:\text{odd}}^{8}(x - 2h \pm h_3 \pm h_4 \pm h_5 \pm h_6)^2\,,
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
\mathcal{M}_{4+}:\ & D(x) = D'_W(x) D_W(x)\,, \\
& D_W(x) = D'_W(x) \text{ in } \mathcal{M}_{3+}\,, \\
& D'_W(x) = D_W(x) \text{ in } \mathcal{M}_{3+}\,.
\end{aligned} \tag{3.43}$$

$\mathcal{N}_{3+}$ and $\mathcal{N}_{4+}$ cases are obtained by setting $h = 0$ in the $\mathcal{M}_{3+}$ and $\mathcal{M}_{4+}$ cases, respectively. From (3.41), (3.42), (3.43), it can be shown that all h_k $(k = 3,4,5,6)$ in the so(8) part have to be zero in all above cases to preserve the positivity condition. Therefore, unlike the [I-1] case, no further fraction of survived supersymmetry can be obtained by adding the parameters h_k.

[II] Other so(10, 2) cases

Adding to the types in [I], there arise two types in the classification of so(10, 2) which contain indecomposable types with the dimension higher than four, that is, ${}^{(\epsilon_D,\epsilon_C)}\Delta_2^-(ih,-ih)$ with $\dim\Delta = 6$ and $\mathrm{ind}\,\Delta = 2$, and ${}^{(+,\epsilon_C)}\Delta_4^+(0)$ with $\dim\Delta = 5$ and $\mathrm{ind}\,\Delta = 2$.

[II-1] $${}^{(\epsilon_D)}\mathcal{F}_1 = {}^{(\epsilon_D,+)}\Delta_2^-(ih,-ih) + \sum_{k=2}^{4}\Delta_0^+(ih_k,-ih_k)$$

The type $\Delta_2^-(ih,-ih)$ under O(10, 2) splits into four distinct parts under the action of $\mathrm{SO}(10,2)_0$. For the C- and D-splittings, we use the following 6×6 matrices,

$$C = \begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \qquad D = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \tag{3.44}$$

which act on a representative of $\Delta_2^-(ih,-ih)$ given in (A.8) and yield the representative ${}^{(\epsilon_D,\epsilon_C)}Z$ of the type ${}^{(\epsilon_D,\epsilon_C)}\Delta_2^-(ih,-ih)$ with respect to the (4+2)D flat metric $\eta = (-,-,+,+,+,+)$,

$${}^{(\epsilon_D,\epsilon_C)}Z = \begin{pmatrix} 0 & -h & 1/\sqrt{2} & 0 & 0 & 0 \\ h & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & -h & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & h & 0 & 0 & \epsilon_D/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 0 & 0 & -\epsilon_D h \\ 0 & 0 & 0 & -\epsilon_D/\sqrt{2} & \epsilon_D h & 0 \end{pmatrix}, \tag{3.45}$$

where $\epsilon_C = +1\,(-1)$ corresponds to the range $h > 0\,(< 0)$, showing the $\epsilon_C = +1$ case to be compatible with the positivity condition. Arranging the matrix ${}^{(\epsilon_D,+)}Z$ and $ih_k\sigma_2$ $(k = 2,3,4)$ on the diagonal part, we obtain a representative of the type ${}^{(\epsilon_D)}\mathcal{F}_1$ with respect to our (10+2)D flat metric η_{AB}. The characteristic polynomial for the type is

$$\begin{aligned} D(x) &= D'_W(x)D_W(x)\,, \qquad D_W(x) = D_{\rho=-1}(x)\,, \qquad D'_W(x) = D_{\rho=1}(x)\,, \\ D_\rho &= \prod_{-:\text{even}}^{4}(x - 3h + \rho\epsilon_D(\pm h_2 \pm h_3 \pm h_4))^2((x - h - \rho\epsilon_D(\pm h_2 \pm h_3 \pm h_4))^2 \times \\ &\quad \times (x + h + \rho\epsilon_D(\pm h_2 \pm h_3 \pm h_4)) - 4x)^2. \end{aligned} \tag{3.46}$$

It can be shown that, for any values of h, h_2, h_3 and h_4, there appears at least one negative eigenvalue from the cubic polynomial part of the parameter x in the last line of (3.46). Therefore the type ${}^{(\epsilon_D)}\mathcal{F}_1$ can not be used to define anti-commutators among supercharges.

[II-2] $\qquad \mathcal{F}_2 = {}^{(+,+)}(\Delta_4^+(0) + \Delta_0^+(0)) + \sum_{k=2}^{4} \Delta_0^+(ih_k, -ih_k)$

The type $\Delta_4^+(0) + \Delta_0^+(0)$ under $\mathrm{O}(10,2)$ splits into two parts ${}^{(+,\epsilon_C)}(\Delta_4^+(0) + \Delta_0^+(0))$ under $\mathrm{SO}(10,2)_0$. A representative in $\Delta_4^+(0)$ is constructed in (A.18), which gives, after the C-slipping generated by the C matrix in (3.44), the representative ${}^{(\epsilon_C)}Z$ in ${}^{(+,\epsilon_C)}(\Delta_4^+(0) + \Delta_0^+(0))$ with respect to the (4+2)D metric $\eta = (-,-,+,+,+,+)$,

$$ {}^{(\epsilon_C)}Z = \frac{\epsilon_C}{2} \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.47} $$

showing $\epsilon_C = -1$ to be forbidden by the positivity condition. We obtain a representative of the type $\mathcal{F}_2$ with respect to the (10+2)D flat metric η_{AB} by arranging the matrix ${}^{(+)}Z$ and $ih_k\sigma_2$ $(k = 2,3,4)$ on the diagonal block. The characteristic polynomial in this $\mathcal{F}_2$ case becomes

$$ \begin{aligned} D(x) &= (D_W(x))^2, \\ D_W(x) &= \prod_{-:\mathrm{even}}^{4} ((x^2 - 2 - (\pm h_2 \pm h_3 \pm h_4)^2)(x(x-2) - (\pm h_2 \pm h_3 \pm h_4)^2) - \\ &\qquad - 2(\pm h_2 \pm h_3 \pm h_4)^2)^2. \end{aligned} \tag{3.48} $$

As in the [II-1] case, for any values of h_2, h_3 and h_4, we have at least one negative eigenvalue as a solution of the fourth order equation of the parameter x in the RHS of (3.48). Hence the type $\mathcal{F}_2$ can not be used for the definition of anti-commutators among supercharges.

With the above two cases, the classification of the $\mathrm{so}(10,2)$ algebra under $\mathrm{SO}(10,2)_0$ has been completed and all possible BPS states arising from the central term Z_{AB} have been exhausted.

4. Conclusion

In summary, we have studied the $N = (1,1)$ non-Poincaré superalgebra in (10+2)D with tensorial central terms $Z^{(k)}$ $(k = 2,3,6,7,10,11)$. The Weyl-projected form of the algebra, i.e. the $N = (1,0)$ superalgebra, has been shown to be reduced to the (10+1)D M-algebra by a timelike dimensional reduction, just as the M-algebra is reduced to the (9+1)D type-IIA algebra by a spacelike dimensional reduction. Another $\mathrm{SO}(9,1)$ covariant projection of the $N = (1,1)$ algebra with dimensional reduction to (9+1)D has also been demonstrated to yield the type-IIB algebra.

From the complete classification of the second-rank central term, we have exhausted all possible BPS states arising from this term. A 1/2-susy BPS state is associated with the semisimple $\mathcal{S}_2$ type in $\mathrm{so}(2,2)$ and is naturally interpreted as a super (2+2)-brane, which is dimensionally reduced to a membrane in (10+1)D and type-IIA and -IIB fundamental strings in (9+1)D. Further fractions less than 1/2 of surviving supersymmetries have been obtained from the extension of the $\mathcal{S}_2$ type to the canonical type in $\mathrm{so}(10,2)$. There has arisen another 1/2-susy BPS state from the nilpotent $\mathcal{N}_{1+}$ type in $\mathrm{so}(2,2)$, which would be observed as an M9 brane in M-theory. The nilpotent type has been caused due to the non-compact property of the group $\mathrm{SO}(10,2)$. Thus the further investigation of the type might give some insight or constraint on the number of time directions of spacetime. We also have obtained the fraction 1/4 of surviving supersymmetries in $\mathcal{M}_{3+}$ and $\mathcal{M}_{4+}$ types and the fraction 3/4 in $\mathcal{N}_{3+}$ and $\mathcal{N}_{4+}$ types. As discussed in section 3.3.2 [III], if BPS states with these fractions of supersymmetry can possibly exist in M-theory, they would be realized as some composite states of M2 and M9 branes. In recent paper [26], it has been shown that a configuration of an M2 brane intersecting two M5 branes also preserves 1/4 or 3/4 of supersymmetry when the product of all three brane charges is negative. It remains to investigate whether there exist (10+1)D supergravity solutions corresponding to those 1/4 and 3/4-susy BPS states, or not.

Besides the second-rank central term, the consideration of other central terms leads us to other various brane configurations inherent in the (10+2)D superalgebra. Among them, the sixth-rank central term $Z^{(6)}$ represents a super (6+2)-brane, which is dimensionally reduced to an M5 brane in M-theory and an NS5 brane in the type-IIB theory. It has been discussed that the worldvolume theory of the (2+2)-brane moving in (10+2)D spacetime is given by (2+2)D self-dual Yang-Mills and gravitational theories [9]. It is intriguing to study whether the (6+2)-brane is described by some (6+2)D integrable systems like the self-dual theories. A candidate would be a (6+2)D version of the 8D self-dual Yang-Mills theory [27] and its gravitational analogue [28].

The method used in this paper to classify the central term $Z^{(2)}_{AB}$ under the $\mathrm{SO}(10,2)$ rotation is also applicable to any second-rank central term in any dimension and signature of spacetime. An example is the $N = (2,0)$ chiral superalgebra in (10+2)D, which is a non-Poincaré type algebra containing central terms $Z^{(2)}$, Z, $Z^{(4)}$ and self-dual ${}^{(+)}Z^{(6)}$ [6],

$$\begin{aligned}\{Q_i^\alpha, Q_j^\beta\} = (\tau^k)_{ij}((\Gamma^{AB}C^{-1})^{\alpha\beta} Z^{(2)}_{k\,AB} + (\Gamma^{A_1\cdots A_6}C^{-1})^{\alpha\beta\,(+)}Z^{(6)}_{k\,A_1\cdots A_6}) + \\ +\epsilon_{ij}((C^{-1})^{\alpha\beta} Z + (\Gamma^{A_1\cdots A_4}C^{-1})^{\alpha\beta} Z^{(4)}_{A_1\cdots A_4}), \qquad (4.1)\end{aligned}$$

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where Q_i^α ($i = 1, 2$) are Majorana-Weyl supercharges with the same chirality and $\tau^k = (\sigma_3, \sigma_1, I_2)$. As noted in the introduction, like the $N = (1, 1)$ case, the $N = (2, 0)$ algebra also reduces to M-, type-IIA and -IIB algebras by dimensional reduction. It is interesting to find what types of BPS states exist in this chiral algebra and to clarify the connection of them with various BPS states in membrane and string theories.

Acknowledgments

I thank N. Kawanaka for useful suggestions. I am grateful to H. Kihara and D.B. Fairlie for discussions and helpful comments. I also thank the department of Mathematical Sciences in University of Durham and the department of Mathematics in King's College, London for their hospitality.

A. Representatives for indecomposable types in $\mathrm{o}(10, 2)$

According to the description of indecomposable types in section 3.2.2, we construct explicit forms of representatives for the types Δ_m relevant to our $\mathrm{o}(10, 2)$ case. Let $(Z, V) \in \Delta_m$ and Z be decomposed as the sum of semisimple S and nilpotent N elements, $Z = S + N$, with $N^{m+1} = 0$. The vector space V can be written as $V = F + NF + \cdots + N^m F$, where F is the subspace of V spanned by eigenvectors of S. From (3.4), the metric $\tau_m(u, v) = \tau(u, N^m v)$ for $u, v \in F$ is symmetric for $m =$ even and anti-symmetric for $m =$ odd. For later use, we define the symmetric, nondegenerate and real bilinear form θ_m as

$$\begin{aligned} \theta_m(u, v) &= \tau_m(u, v) && \text{if } m = \text{even}\,, \\ &= \tau_m(u, Sv) && \text{if } m = \text{odd and } S \neq 0\,. \end{aligned} \tag{A.1}$$

A.1 $\Delta_m^\epsilon(ih, -ih)$

We can choose a pair of vectors $\{e, f\}$ spanning the subspace F of V such that $Se = -hf$ and $Sf = he$ ($h \in \mathbb{R}$). From $\theta_m(Su, v) + \theta_m(u, Sv) = 0$, we have $\theta_m(e, f) = 0$ while $\theta_m(e, e) = \theta_m(f, f)$. We have two choices of the sign of $\theta_m(e, e)$, which is labeled by the superscript $\epsilon = \pm 1$ in the type.

A.1.1 $m = 0$ $\dim \Delta = 2$

Since $N^{0+1} = 0$, this type contains only a semisimple element $Z = S$. The form of S on $V = F = \{e, f\}$ with respect to (w.r.t.) the metric τ becomes

$$Z = S = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}, \qquad \text{w.r.t.} \qquad \tau = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{A.2}$$

The form of the metric shows that the type Δ^+ has the index 0, while Δ^- has the index 2.

A.1.2 $m=1$ $\dim\Delta=4$, $\mathrm{ind}\,\Delta=2$

Adding to a semisimple element S, we have a nilpotent N with $N^2=0$. A representative in this case is given as

$$Z=S+N\,,\quad S=h\begin{pmatrix}-\varepsilon_2 & 0\\ 0 & -\varepsilon_2\end{pmatrix},\qquad N=\begin{pmatrix}0&0\\ I_2&0\end{pmatrix}\tag{A.3}$$

w.r.t. the metric

$$\tau=\frac{\epsilon}{h}\begin{pmatrix}0&\varepsilon_2\\ -\varepsilon_2&0\end{pmatrix}.\tag{A.4}$$

We perform a transformation of V onto V' by an isomorphism ϕ noted in section 3.2.2 to make the form of the metric τ to be $\tau'=\phi\tau\tilde{\phi}=(-,-,+,+)$. With a realization O_ϕ of ϕ, we have

$$\begin{aligned}Z'=\tilde{O}_\phi Z O_\phi&=\frac{1}{2}\begin{pmatrix}-(2h+1)\varepsilon_2 & \varepsilon_2\\ -\varepsilon_2 & -(2h-1)\varepsilon_2\end{pmatrix},\\ O_\phi&=\sqrt{\frac{h}{2}}\begin{pmatrix}I_2&-I_2\\ \varepsilon_2&\varepsilon_2\end{pmatrix},\end{aligned}\tag{A.5}$$

where $h>0$ ($h<0$) corresponds to $\epsilon=+1\,(-1)$.

A.1.3 $m=2$ $\epsilon=-1$, $\dim\Delta=6$, $\mathrm{ind}\,\Delta=2$

Since the $\epsilon=+1$ case has $\mathrm{ind}\,\Delta=4$, only the $\epsilon=-1$ case is relevant to our o$(10,2)$ algebra. The form of a representative in the $\epsilon=-1$ case becomes

$$Z=S+N\,,\qquad S=h\begin{pmatrix}-\varepsilon_2&0&0\\ 0&-\varepsilon_2&0\\ 0&0&-\varepsilon_2\end{pmatrix},\qquad N=\begin{pmatrix}0&0&0\\ I_2&0&0\\ 0&I_2&0\end{pmatrix},\tag{A.6}$$

w.r.t. the metric

$$\tau=\begin{pmatrix}0&0&-I_2\\ 0&I_2&0\\ -I_2&0&0\end{pmatrix}.\tag{A.7}$$

Performing the ϕ-transformation O_ϕ, we have a new representative

$$\begin{aligned}Z'=\tilde{O}_\phi Z O_\phi&=\frac{1}{\sqrt{2}}\begin{pmatrix}-\sqrt{2}h\varepsilon_2&I_2&0\\ I_2&-\sqrt{2}h\varepsilon_2&I_2\\ 0&-I_2&-\sqrt{2}h\varepsilon_2\end{pmatrix},\\ O_\phi&=\frac{1}{\sqrt{2}}\begin{pmatrix}I_2&0&I_2\\ 0&\sqrt{2}I_2&0\\ I_2&0&-I_2\end{pmatrix},\end{aligned}\tag{A.8}$$

w.r.t. $\tau'=(-,-,+,+,+,+)$.

A.2 $\Delta_m(h,-h)$

As eigenvalues of S are real, we can take eigenvectors $\{e,f\}$ such that $Se = he$ and $Sf = -hf$. We can derive $\theta_m(e,e) = \theta_m(f,f) = 0$ and, without restriction, take $\{e,f\}$ to satisfy $\theta_m(e,f) = 1$.

A.2.1 $m = 0$ $\dim\Delta = 2$, $\operatorname{ind}\Delta = 1$

This case contains only a semisimple element S w.r.t. the metric τ,

$$S = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}, \qquad \text{w.r.t. } \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.9}$$

By the ϕ-transformation O_ϕ, we have

$$S' = \tilde{O}_\phi S O_\phi = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}, \qquad O_\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{A.10}$$

w.r.t. $\tau' = (-,+)$.

A.2.2 $m = 1$ $\dim\Delta = 4$, $\operatorname{ind}\Delta = 2$

This case contains a nilpotent N with $N^2 = 0$, adding to a semisimple S,

$$Z = S + N\,, \qquad S = h\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ I_2 & 0 \end{pmatrix}, \tag{A.11}$$

w.r.t. the metric

$$\tau = \begin{pmatrix} 0 & -\varepsilon_2 \\ \varepsilon_2 & 0 \end{pmatrix}. \tag{A.12}$$

By the ϕ-transformation O_ϕ, we obtain another representative

$$\begin{aligned} Z' = \tilde{O}_\phi Z O_\phi &= \frac{1}{2}\begin{pmatrix} \varepsilon_2 & 2h\sigma_3 + \varepsilon_2 \\ 2h\sigma_3 - \varepsilon_2 & -\varepsilon_2 \end{pmatrix}, \\ O_\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & I_2 \\ -\varepsilon_2 & \varepsilon_2 \end{pmatrix}, \end{aligned} \tag{A.13}$$

w.r.t. $\tau' = (-,-,+,+)$.

A.3 $\Delta_m^\epsilon(0)$

In this case, there appears only a nilpotent N with $N^{m+1} = 0$. Let e be a vector in V, then we have $V = e + Ne + \cdots + N^m e$. We can take $\theta_m(e,e) = \epsilon$ without any restriction, which determines the form of the metric completely.

A.3.1 $m = 0$ $\dim\Delta = 1$

This case contains the trivial element $Z = (0)$ w.r.t. the metric $\tau = \epsilon$.

A.3.2 $m = 2$ $\dim\Delta = 3$

In this case, we have a 3×3 nilpotent matrix

$$Z = N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{w.r.t. } \tau = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & -\epsilon & 0 \\ \epsilon & 0 & 0 \end{pmatrix}. \tag{A.14}$$

In the $\epsilon = +1$ case, we obtain a representative in $\Delta_2^+(0)$ w.r.t. $\tau' = (-,-,+)$ by the ϕ-transformation O_ϕ,

$$\begin{aligned} N' = \tilde{O}_\phi N O_\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ O_\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{A.15}$$

On the other hand, in the $\epsilon = -1$ case, by another ϕ-transformation O'_ϕ, we have a representative in $\Delta_2^-(0)$,

$$\begin{aligned} N' = \tilde{O}'_\phi N O'_\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\ O'_\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}, \end{aligned} \tag{A.16}$$

w.r.t. $\tau' = (-,+,+)$.

A.3.3 $m = 4$ $\epsilon = +1$**,** $\dim\Delta = 5$**,** $\mathrm{ind}\,\Delta = 2$

The case with $\epsilon = -1$ is forbidden since it has $\mathrm{ind}\,\Delta = 3$. A representative for the $\epsilon = +1$ case is given as

$$Z = N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \text{w.r.t. } \tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.17}$$

Performing the ϕ-transformation O_ϕ, we obtain another representative in $\Delta_4^+(0)$,

$$N' = \tilde{O}_\phi N O_\phi = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & \sqrt{2} & 1 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -\sqrt{2} & 1 & 0 \end{pmatrix},$$

$$O_\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{A.18}$$

w.r.t. $\tau' = (-,-,+,+,+)$.

A.4 $\Delta_1(0,0)$ $\dim\Delta = 4$**,** $\operatorname{ind}\Delta = 2$

This type gives a nilpotent $Z = N$ with $N^2 = 0$. Let us take two vectors $\{e, f\}$ in V as eigenvectors of S, then the vector space V can be spanned by $\{e, f, Ne, Nf\}$. We can derive $\theta_1(e,e) = \theta_1(f,f) = 0$ and take $\theta_1(e,f)$ equal to $+1$. The representative in $\Delta_1(0,0)$ and the metric τ derived from the setup are the same as those in (A.11), (A.12). Using the ϕ-transformation O_ϕ in (A.13), we obtain another representative w.r.t. the metric $\tau' = (-,-,+,+)$,

$$N' = \tilde{O}_\phi N O_\phi = \frac{1}{2} \begin{pmatrix} \varepsilon_2 & \varepsilon_2 \\ -\varepsilon_2 & -\varepsilon_2 \end{pmatrix}. \tag{A.19}$$

A.5 $\Delta_0(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$ $\dim\Delta = 4$**,** $\operatorname{ind}\Delta = 2$

We have a semisimple $Z = S$ with eigenvalues $\{\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta}\}$. Let $\zeta = h_1 + ih_2$ and $\{e_\pm, f_\pm\}$ be vectors in V satisfying

$$\begin{aligned} Se_\pm &= \pm(h_1 e_\pm - h_2 f_\pm), \\ Sf_\pm &= \pm(h_1 f_\pm + h_2 e_\pm). \end{aligned} \tag{A.20}$$

The values of θ_0 for $\{e_\pm, f_\pm\}$ are given as

$$\begin{aligned} \theta_0(e_\pm, e_\pm) &= \theta_0(f_\pm, f_\pm) = 0\,, \\ \theta_0(e_\pm, f_\pm) &= \theta_0(e_\pm, f_\mp) = 0\,, \\ \theta_0(e_\pm, e_\mp) &= -\theta_0(f_\pm, f_\mp) = 1\,. \end{aligned} \tag{A.21}$$

Explicit forms of S and the metric τ w.r.t. $\{e_\pm, f_\pm\}$ are

$$S = \begin{pmatrix} h_1\sigma_3 & h_2\sigma_3 \\ -h_2\sigma_3 & h_1\sigma_3 \end{pmatrix}, \qquad \text{w.r.t. } \tau = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{A.22}$$

By the ϕ-transformation O_ϕ, we have another representative

$$\begin{aligned} S' = \tilde{O}_\phi S O_\phi &= \begin{pmatrix} -h_2\varepsilon_2 & h_1 I_2 \\ h_1 I_2 & -h_2\varepsilon_2 \end{pmatrix}, \\ O_\phi &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \end{aligned} \tag{A.23}$$

w.r.t. $\tau' = (-,-,+,+)$.

Having obtained the above representatives in indecomposable types, we arrange them on the diagonal part of 12×12 matrix in order to obtain $\mathrm{o}(10,2)$ representatives with $\mathrm{ind}\,\Delta = 2$. In general, we have to transform the obtained representatives into the other forms with respect to our (10+2)D metric $\eta = (-,-,+,\ldots,+)$. Here we give an example in the $\mathrm{so}(2,2)$ case in section 3.3.1. Let us take the sum $\Delta = \Delta_0(h_1,-h_1)+\Delta_0(h_2,-h_2)$, which has $\dim\Delta = 4$ and $\mathrm{ind}\,\Delta = 2$. From section A.2.1, we have a representative Z for the type Δ w.r.t. the metric $\tau = (-,+,-,+)$,

$$Z = \begin{pmatrix} h_1\sigma_1 & 0 \\ 0 & h_2\sigma_1 \end{pmatrix}. \tag{A.24}$$

Then, by the ϕ-transformation O_ϕ, we have another representative in Δ w.r.t. our (2+2)D metric $\tau' = (-,-,+,+)$,

$$\begin{aligned} Z' = \tilde{O}_\phi Z O_\phi &= \begin{pmatrix} 0 & 0 & h_1 & 0 \\ 0 & 0 & 0 & h_2 \\ h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \end{pmatrix}, \\ O_\phi &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{A.25}$$

which corresponds to the $\mathcal{S}_1$ case in section 3.3.1.

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